Assignment on Duality and Sensitivity Analysis

6.1-15. For any linear programming problem in our standard form and its dual problem, label each of the following statements as true or false and then justify your answer.

(a) The sum of the number of functional constraints and the number of variables (before augmenting) is the same for both the primal and the dual problems.

(b) At each iteration, the simplex method simultaneously identifies a CPF solution for the primal problem and a CPF solution for the dual problem such that their objective function values are the same.

(c) If the primal problem has an unbounded objective function, then the optimal value of the objective function for the dual problem must be zero.

6.3-7.* Reconsider the model of Prob. 6.1-4*b*.

(a) Construct its dual problem.

(b) Solve this dual problem graphically.

(c) Use the result from part (*b*) to identify the nonbasic variables and basic variables for the optimal BF solution for the primal problem.

(d) Use the results from part (*c*) to obtain the optimal solution for the primal problem directly by using Gaussian elimination to solve for its basic variables, starting from the initial system of equations [excluding Eq. (0)] constructed for the simplex method and setting the nonbasic variables to zero.

(e) Use the results from part (*c*) to identify the defining equations (see Sec. 5.1) for the optimal CPF solution for the primal problem, and then use these equations to find this solution.

6.4-1. Consider the following problem.

Maximize $\quad Z = x_1 + x_2,$

subject to

$$x_1 + 2x_2 = 10$$
$$2x_1 + x_2 \geq 2$$

and

$$x_2 \geq 0 \qquad (x_1 \text{ unconstrained in sign}).$$

(a) Use the SOB method to construct the dual problem.

(b) Use Table 6.12 to convert the primal problem to our standard form given at the beginning of Sec. 6.1, and construct the corresponding dual problem. Then show that this dual problem is equivalent to the one obtained in part (*a*).

6.4-2. Consider the primal and dual problems in our standard form presented in matrix notation at the beginning of Sec. 6.1. Use only this definition of the dual problem for a primal problem in this form to prove each of the following results.

(a) If the functional constraints for the primal problem $\mathbf{Ax} \leq \mathbf{b}$ are changed to $\mathbf{Ax} = \mathbf{b}$, the only resulting change in the dual problem is to *delete* the nonnegativity constraints, $\mathbf{y} \geq \mathbf{0}$. (*Hint:* The constraints $\mathbf{Ax} = \mathbf{b}$ are equivalent to the set of constraints $\mathbf{Ax} \leq \mathbf{b}$ *and* $\mathbf{Ax} \geq \mathbf{b}$.)

(b) If the functional constraints for the primal problem $\mathbf{Ax} \leq \mathbf{b}$ are changed to $\mathbf{Ax} \geq \mathbf{b}$, the only resulting change in the dual problem is that the nonnegativity constraints $\mathbf{y} \geq \mathbf{0}$ are replaced by nonpositivity constraints $\mathbf{y} \leq \mathbf{0}$, where the current dual variables are interpreted as the negative of the original dual variables. (*Hint:* The constraints $\mathbf{Ax} \geq \mathbf{b}$ are equivalent to $-\mathbf{Ax} \leq -\mathbf{b}$.)

(c) If the nonnegativity constraints for the primal problem $\mathbf{x} \geq \mathbf{0}$ are deleted, the only resulting change in the dual problem is to replace the functional constraints $\mathbf{yA} \geq \mathbf{c}$ by $\mathbf{yA} = \mathbf{c}$. (*Hint:* A variable unconstrained in sign can be replaced by the difference of two nonnegative variables.)

6.6-1.* Consider the following problem.

$$\text{Maximize} \quad Z = 3x_1 + x_2 + 4x_3,$$

subject to

$$6x_1 + 3x_2 + 5x_3 \le 25$$
$$3x_1 + 4x_2 + 5x_3 \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

The corresponding final set of equations yielding the optimal solution is

$$(0) \quad Z + 2x_2 + \frac{1}{5}x_4 + \frac{3}{5}x_5 = 17$$

$$(1) \quad x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3}x_5 = \frac{5}{3}$$

$$(2) \quad x_2 + x_3 - \frac{1}{5}x_4 + \frac{2}{5}x_5 = 3.$$

(a) Identify the optimal solution from this set of equations.
(b) Construct the dual problem.

(c) Identify the optimal solution for the dual problem from the final set of equations. Verify this solution by solving the dual problem graphically.

(d) Suppose that the original problem is changed to

$$\text{Maximize} \qquad Z = 3x_1 + 3x_2 + 4x_3,$$

subject to

$$6x_1 + 2x_2 + 5x_3 \le 25$$
$$3x_1 + 3x_2 + 5x_3 \le 20$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

Use duality theory to determine whether the previous optimal solution is still optimal.

(e) Use the fundamental insight presented in Sec. 5.3 to identify the new coefficients of x_2 in the final set of equations after it has been adjusted for the changes in the original problem given in part (*d*).

(f) Now suppose that the only change in the original problem is that a new variable x_{new} has been introduced into the model as follows:

$$\text{Maximize} \quad Z = 3x_1 + x_2 + 4x_3 + 2x_{new},$$

subject to

$$6x_1 + 3x_2 + 5x_3 + 3x_{new} \le 25$$
$$3x_1 + 4x_2 + 5x_3 + 2x_{new} \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_{new} \ge 0.$$

Use duality theory to determine whether the previous optimal solution, along with $x_{new} = 0$, is still optimal.

(g) Use the fundamental insight presented in Sec. 5.3 to identify the coefficients of x_{new} as a nonbasic variable in the final set of equations resulting from the introduction of x_{new} into the original model as shown in part (*f*).

D,I **6.7-1.*** Consider the following problem.

$$\text{Maximize} \quad Z = -5x_1 + 5x_2 + 13x_3,$$

subject to

$$-x_1 + x_2 + 3x_3 \le 20$$
$$12x_1 + 4x_2 + 10x_3 \le 90$$

and

$$x_j \ge 0 \quad (j = 1, 2, 3).$$

If we let x_4 and x_5 be the slack variables for the respective constraints, the simplex method yields the following final set of equations:

$$
\begin{aligned}
&(0) \quad Z \qquad\qquad + 2x_3 + 5x_4 \qquad = 100 \\
&(1) \quad -x_1 + x_2 + 3x_3 + x_4 \qquad = 20 \\
&(2) \quad 16x_1 \qquad - 2x_3 - 4x_4 + x_5 = 10.
\end{aligned}
$$

Now you are to conduct sensitivity analysis by *independently* investigating each of the following nine changes in the original model. For each change, use the sensitivity analysis procedure to revise this set of equations (in tableau form) and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. (Do not reoptimize.)

(a) Change the right-hand side of constraint 1 to

$$b_1 = 30.$$

(b) Change the right-hand side of constraint 2 to

$$b_2 = 70.$$

(c) Change the right-hand sides to

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 100 \end{bmatrix}.$$

(d) Change the coefficient of x_3 in the objective function to

$c_3 = 8$.

(e) Change the coefficients of x_1 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}.$$

(f) Change the coefficients of x_2 to

$$\begin{bmatrix} c_2 \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}.$$

(g) Introduce a new variable x_6 with coefficients

$$\begin{bmatrix} c_6 \\ a_{16} \\ a_{26} \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix}.$$

(h) Introduce a new constraint $2x_1 + 3x_2 + 5x_3 \leq 50$. (Denote its slack variable by x_6.)

(i) Change constraint 2 to

$$10x_1 + 5x_2 + 10x_3 \leq 100.$$